

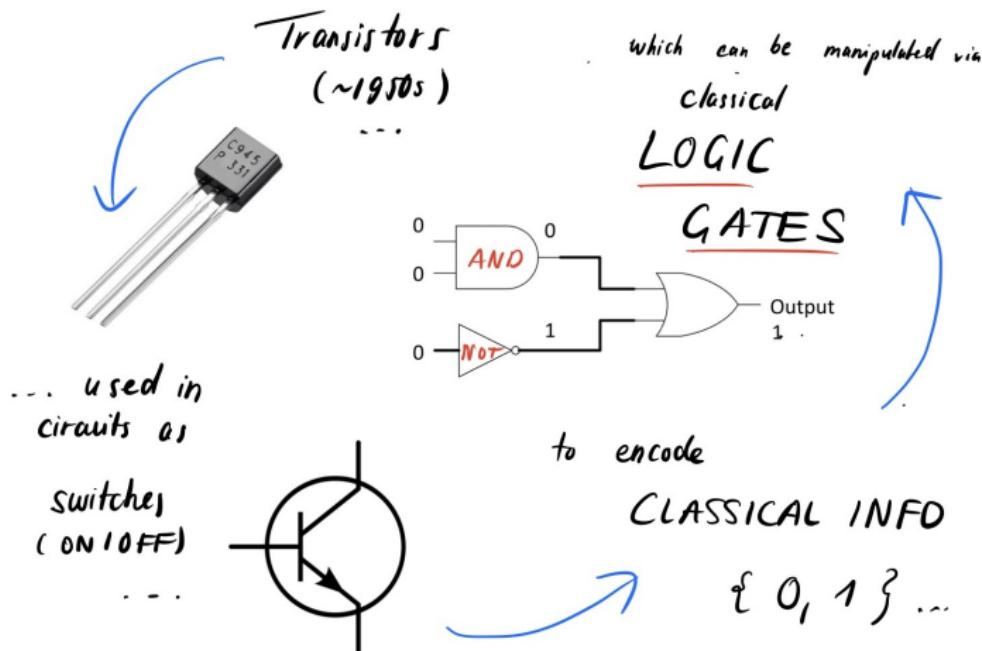
Many facets of quantum magic

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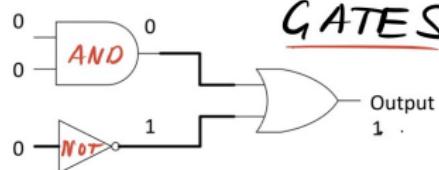
Classical Computation



which can be manipulated via classical

LOGIC

GATES

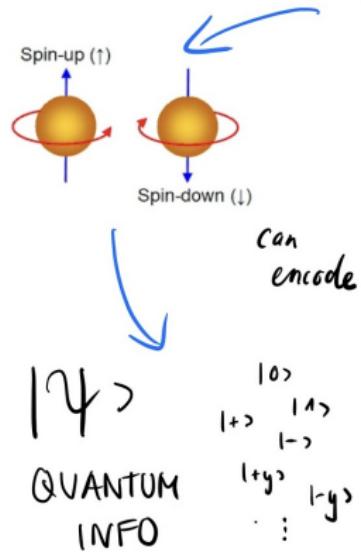


to encode

CLASSICAL INFO

{ 0, 1 } ...

Quantum Computation

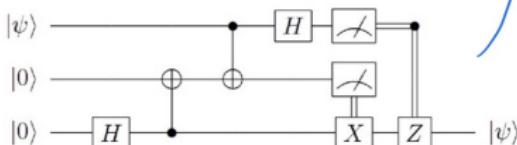


Quantum systems
(now!)

... quantum LOGIC GATES

$$\text{e.g. } H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$|10\rangle \mapsto \frac{1}{\sqrt{2}} (|10\rangle + |11\rangle)$$
$$|11\rangle \mapsto \frac{1}{\sqrt{2}} (|10\rangle - |11\rangle)$$



which can be manipulated
via ...

Universal Gate Sets

UNIVERSAL GATE SET

any classical / quantum operation can be expressed
as finite sequence of gates from this set

classical

$$\{ \text{AND}, \text{NOT} \}$$

quantum

$$\{ H, S, CNOT, T \}$$

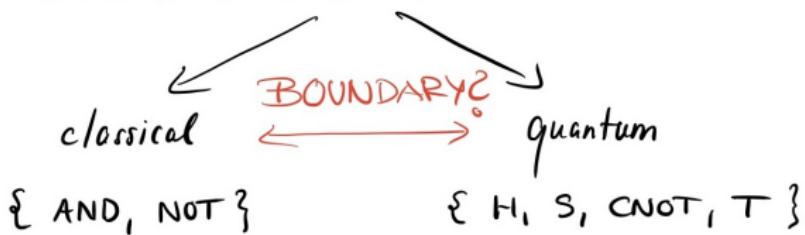
$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

The Boundary?

UNIVERSAL GATE SET

any classical / quantum operation can be expressed as finite sequence of gates from this set



$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{i\pi}{4}} \end{pmatrix}$$

Classical Simulation

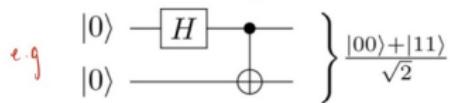
Clifford

{ H, S, CNOT }

no T-gate

~1 efficiently simulable
on CLASSICAL computer

! but ENTANGLEMENT
possible



Magic States

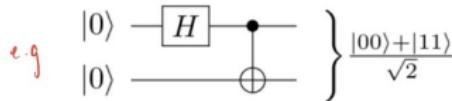
Clifford gates

$$\{ H, S, CNOT \}$$

no T-gate

~1 efficiently simulable
on CLASSICAL computer

! but ENTANGLEMENT
possible



Clifford gates,
+ magical states

$$|0^n\rangle \xrightarrow{\neq C} |\text{mag}\rangle$$

Measures of magic

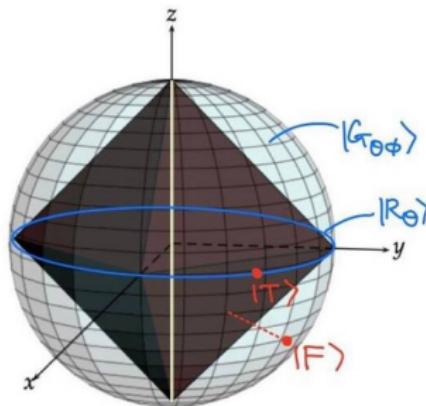


Figure: Stabilizer state octahedron in the Bloch sphere ($n = 1$)

Now, to higher dimensions:

$$|\text{STAB}_n| = 2^{(1/2+o(1))n^2}$$

Measures of magic

Type	Definition	Cost	Simulation?
Distance based	$F(\psi\rangle) = \max_{\rho} \langle \rho \psi \rangle ^2$	$ \text{STAB}_n \sim 2^{O(n^2)}$?
Stabilizer rank	$SR(\psi\rangle) = \min \left\{ \chi : \psi\rangle = \sum_{i=1}^{\chi} c_i \rho_i\rangle \right\}$	# tuples in STAB_n	✓
Pauli Spectrum based	$PS(\psi\rangle) = \left\{ \frac{\langle \psi P_i \psi \rangle}{\langle \psi \psi \rangle} : P_i \in \mathcal{P}_n \right\}$	$ \mathcal{P}_n = 2^{O(n)}$?

$$\mathcal{P}_n = \left\{ \pm P_1 \otimes P_2 \otimes \cdots \otimes P_n : P_i \in \{I, X, Y, Z\} \right\}$$

Measures of Magic

Pauli strings		Stabilizer state $\frac{1}{\sqrt{2}}(00\rangle + 11\rangle)$		Magic state $\frac{1}{\sqrt{2}}(00\rangle + e^{i\pi/4} 11\rangle)$	
II	YI	1	0	1	0
IX	YX	0	0	0	$1/\sqrt{2}$
IY	YY	0	-1	0	$-1/\sqrt{2}$
IZ	YZ	0	0	0	0
XI	ZI	0	0	0	0
XX	ZX	1	0	$1/\sqrt{2}$	0
XY	ZY	0	0	$1/\sqrt{2}$	0
XZ	ZZ	0	1	0	1

Stabilizer Rényi- α entropy: $H_\alpha = \frac{1}{1-\alpha} \log_2 \sum_{i=1}^{4^n} p_i^\alpha$

where $p_i = \frac{1}{2^n} \left(\frac{\langle \psi | P_i | \psi \rangle}{\langle \psi | \psi \rangle} \right)^2$

Pauli Spectrum



Fidelity



Stabilizer rank

Pauli spectrum \iff Stabilizer fidelity

Pauli Spectrum



Fidelity



Stabilizer rank

Pauli spectrum \iff Stabilizer fidelity

Product state

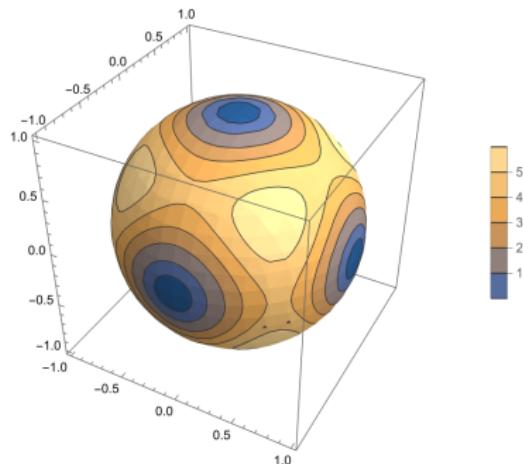


Figure: Rényi-2 entropy of the generalized state $G_{\theta\phi}^{\otimes 10}$

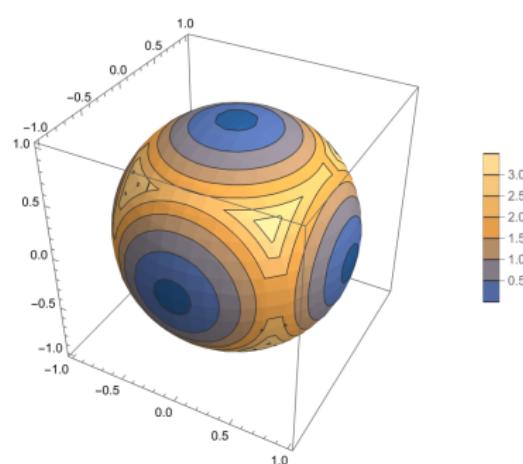


Figure: -log of the fidelity of the generalized state $G_{\theta\phi}^{\otimes 10}$

Pauli spectrum \iff Stabilizer fidelity

Product state

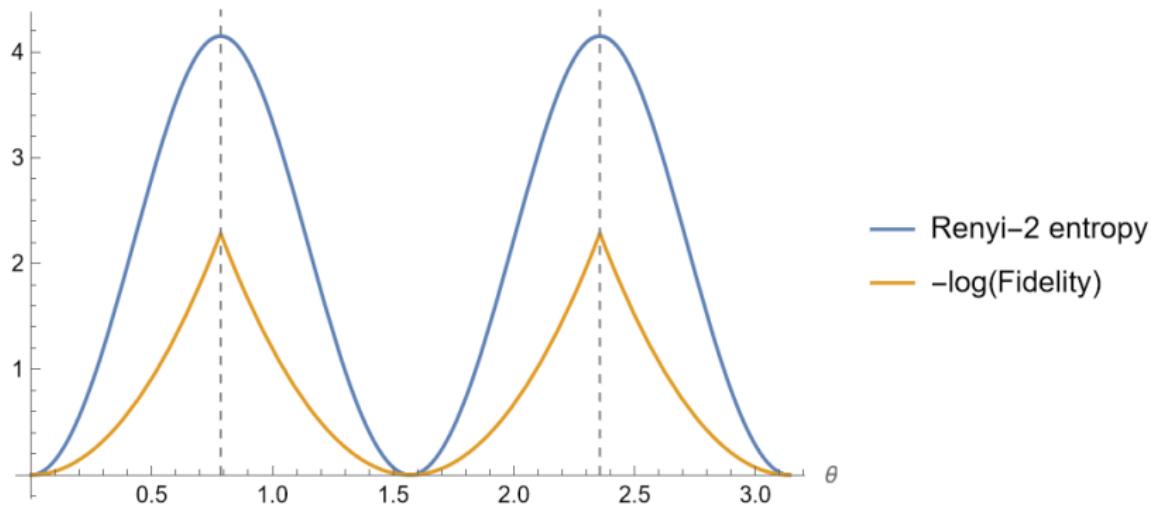


Figure: Rényi-2 entropy and $-\log$ of fidelity for $R_\theta^{\otimes 10}$

Pauli spectrum \iff Stabilizer fidelity

Entangled state

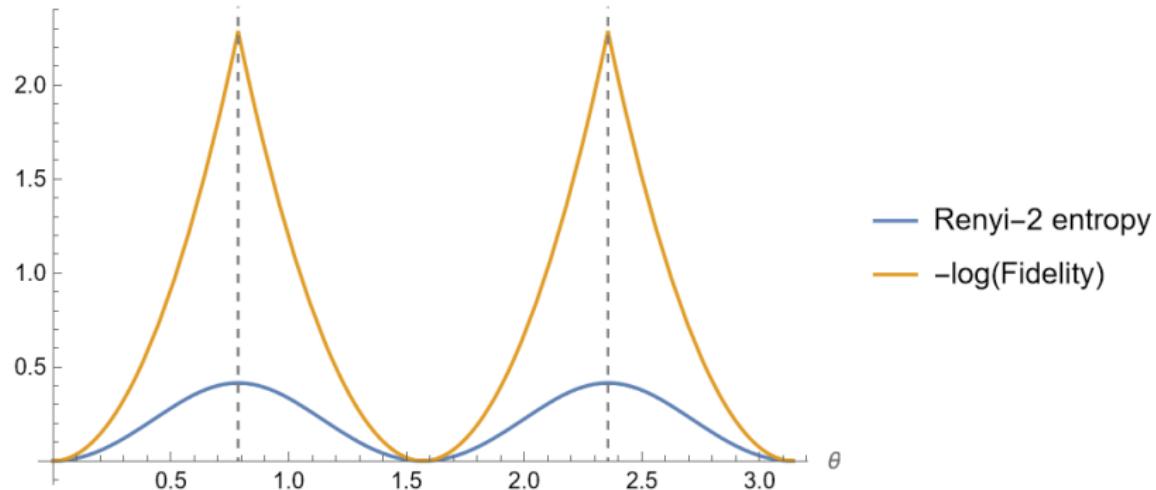


Figure: Rényi-2 entropy and $-\log$ of fidelity for $\frac{1}{\sqrt{2}} \left(|0\rangle^{\otimes 10} + e^{i\theta} |1\rangle^{\otimes 10} \right)$

Pauli spectrum \iff Stabilizer rank

Pauli Spectrum



Fidelity



Stabilizer rank

Pauli spectrum \iff Stabilizer rank

Can we say something about the stabilizer rank
from the Pauli spectrum?

Pauli spectrum \iff Stabilizer rank

$$SR(|\phi\rangle) = \min \left\{ \chi : |\phi\rangle = \sum_{i=1}^{\chi} c_i |\rho_i\rangle \right\}$$

For stabilizer rank 1 states, **YES!**



Pauli spectrum of a stabilizer state only contains 1, 0, -1.

Pauli spectrum \iff Stabilizer rank

What about stabilizer rank 2 states?

Pauli spectrum \iff Stabilizer rank

$$SR(|\phi\rangle) = \min \left\{ \chi : |\phi\rangle = \sum_{i=1}^{\chi} c_i |\rho_i\rangle \right\}$$

Any stabilizer rank 2 state can be put in the canonical form by acting only Clifford operations C :

$$C|\phi\rangle = |\psi\rangle = \frac{1}{K} \left(|0\rangle^{\otimes n} + \gamma |1\rangle^{\otimes a} \otimes |0\rangle^{\otimes b} \otimes |+\rangle^{\otimes n-a-b} \right)$$

Pauli spectrum \iff Stabilizer rank

Clifford operations: take Pauli to Pauli

Clifford operations do **not** change the Pauli Spectrum!

Pauli spectrum \iff Stabilizer rank

$$(17) = \frac{1}{k^2} \left(\frac{1}{k^2} \right)^{\frac{1}{k}} \left[\frac{(k^2)^{\frac{1}{k}} - 1}{(k^2)^{\frac{1}{k}}} \right]$$

$$\begin{aligned} 142 &= \frac{x^2 b}{x^2 a^2} = \frac{1}{a^2} \left(\frac{b}{a} \right)^2 \\ &\quad \text{where } \frac{b}{a} = \sqrt{\frac{142}{100}} = \sqrt{1.42} \\ -xy \cdot 142 &= \frac{x^2 b}{x^2 a^2} = \frac{1}{a^2} \left(\frac{b}{a} \right)^2 = \frac{1}{a^2} \left(\frac{142}{100} \right)^2 = \frac{1}{a^2} \left(\frac{142}{100} \right)^2 \end{aligned}$$

$$\begin{aligned} \text{... } XY \cdot 1 \# &= \begin{cases} X \in \text{set} \\ Y \in \text{set} \end{cases} \quad \text{not } f(1) \\ (x_1 \dots x_n) \# &= \begin{cases} x_1 \in \text{set} \\ \dots \\ x_n \in \text{set} \end{cases} \quad \text{functional} \end{aligned}$$

4

A close-up photograph of a spiral-bound notebook page. The page is ruled with light blue horizontal lines. The spiral binding is visible along the left edge, and the paper has a slightly textured appearance.

We are finding Pauli spectrum of

$$v_p = \frac{1}{k} (1 - \dots - 0^n) + \gamma \frac{1}{k} 10^{b+n-a-b} \begin{cases} & a \in \{0, 1\}, \\ & 0 \leq b \leq n. \end{cases}$$

for $a =$

$$\langle \psi | I | \psi \rangle = 1$$

$$\langle 4 \rangle - z - 14\rangle = \frac{z^{\frac{6}{b}}}{\frac{b}{5}} + 1$$

$\begin{cases} z^{\frac{6}{b}} \\ \text{also if } z = b \end{cases}$
 $\frac{1}{K} \left(1 + \frac{2r \cos \theta}{(\sqrt{2})^n b} \right)$
 $2^n - b$

$$\text{and } \begin{cases} x \in b \\ \text{never } x \in b. \end{cases} \quad \frac{1}{k^2} \left(\frac{2\pi \cos \theta}{(\sqrt{2})^2} \cdot \sqrt{b^2 + r^2} \right) \quad 2^{k-2} - 1$$

$$XZ - 14 = \frac{V\epsilon b}{k_1^2(3)^2 y} \left[\frac{(-1)^n y^2 + 2^* y^*}{(12)^{n+6}} \right] \quad 2^n - 2^{n-1}$$

$$x \in \mathbb{G}, \quad \text{none in } b$$

z_{eb} only $\frac{1}{k} \left(\frac{\text{force}}{(k_2)^{1/2}} + r^2 \right)$ $\frac{z_e}{b} \frac{V/I}{W/b}$ (2^n)
 z_{eb} also in b $\frac{1}{k} z_{eb} C_0 R$

$$K^2 = \frac{(V_2)^{1985}}{\frac{X/2}{L} - \frac{X/2/L}{R_0}} \quad \checkmark$$

complex $\beta = t e^{i\theta}$ $\underline{1}'$ β^*

$$\frac{g}{f(x)} \cdot \frac{x^m - b^m}{x - b}$$

$$\begin{array}{r} 2^b \\ \times 3^a \\ \hline 2^{b-a} - 2^a \end{array}$$

$$\frac{3^{n-b-1}}{2^{n-b-1}}$$

$$\left[\begin{array}{c} i^{\text{N-1}} \\ \vdots \\ i^1 \end{array} \right] = 3^{\text{N-1}} - 2^{\text{N-1}}$$

Pauli spectrum \iff Stabilizer rank

Possible constraints:

1. Number of 0s
2. Number of 1s
3. Sum of all Pauli spectrum elements

Summary

Pauli Spectrum 



Fidelity



Stabilizer rank

