Many facets of quantum magic

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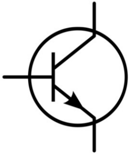
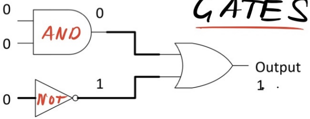
Classical Computation

Transistors (~1950s) ...

... used in circuits as switches (ON/OFF) ...



... which can be manipulated via classical LOGIC GATES

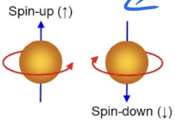


to encode CLASSICAL INFO

{0, 1} ...

Quantum Computation

Quantum systems (now!)



can encode

$|\psi\rangle$

QUANTUM INFO

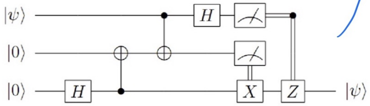
$|0\rangle$ $|1\rangle$ $|+\rangle$ $|-\rangle$ $|+y\rangle$ $|-y\rangle$ ⋮

... quantum LOGIC GATES

e.g. $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$|0\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

$|1\rangle \mapsto \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$



which can be manipulated via ...

Universal Gate Sets

UNIVERSAL GATE SET

↪ any classical / quantum operation can be expressed as <u>finite sequence</u> of gates from this set

classical

{ AND, NOT }

quantum

{ H, S, CNOT, T }

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

The Boundary?

UNIVERSAL GATE SET

↪ any classical / quantum operation can be expressed as finite sequence of gates from this set

classical ⟵ BOUNDARY? ⟶ quantum

classical: $\{\text{AND}, \text{NOT}\}$

quantum: $\{H, S, \text{CNOT}, T\}$

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}$$

Classical Simulation

Clifford

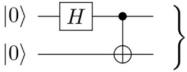
$\{ H, S, CNOT \}$

no T-gate

⇝ efficiently simulable on CLASSICAL computer

! but ENTANGLEMENT possible

e.g



$$\frac{|00\rangle + |11\rangle}{\sqrt{2}}$$

Magic States

Clifford gates

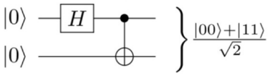
$\{H, S, CNOT\}$

no T-gate

⇝ efficiently simulable on CLASSICAL computer

! but ENTANGLEMENT possible

e.g



Clifford gates + magical states

$|0^n\rangle \xmapsto{\not\in C} |mag\rangle$

Measures of magic

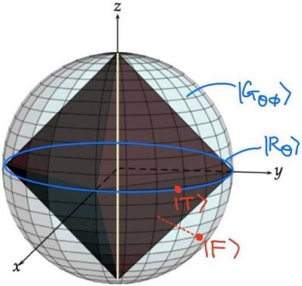


Figure: Stabilizer state octahedron in the Bloch sphere ($n = 1$)

Now, to higher dimensions:

$$|\mathrm{STAB}_n| = 2^{(1/2+o(1))n^2}$$

Measures of magic

Type	Definition	Cost	Simulation?
Distance based	$F(\|\psi\rangle) = \max_{\rho} \|\langle\rho\|\psi\rangle\|^2$	$\|\mathrm{STAB}_n\| \sim 2^{O(n^2)}$	?
Stabilizer rank	$SR(\|\psi\rangle) = \min\left\{\chi : \|\psi\rangle = \sum_{i=1}^{\chi} c_i \|\rho_i\rangle\right\}$	# tuples in STAB_n	✓
Pauli Spectrum based	$PS(\|\psi\rangle) = \left\{\frac{\langle\psi\|P_i\|\psi\rangle}{\langle\psi\|\psi\rangle} : P_i \in \mathcal{P}_n\right\}$	$\|\mathcal{P}_n\| = 2^{O(n)}$	?

$$\mathcal{P}_n = \left\{ \pm P_1 \otimes P_2 \otimes \cdots \otimes P_n : P_i \in \{I, X, Y, Z\} \right\}$$

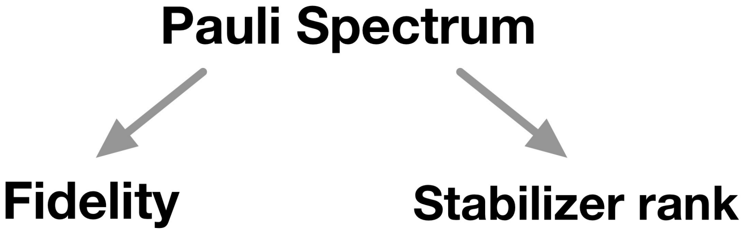
Measures of Magic

Pauli strings		Stabilizer state $\frac{1}{\sqrt{2}}(\|00\rangle + \|11\rangle)$		Magic state $\frac{1}{\sqrt{2}}(\|00\rangle + e^{i\pi/4}\|11\rangle)$	
II	YI	1	0	1	0
IX	YX	0	0	0	$1/\sqrt{2}$
IY	YY	0	-1	0	$-1/\sqrt{2}$
IZ	YZ	0	0	0	0
XI	ZI	0	0	0	0
XX	ZX	1	0	$1/\sqrt{2}$	0
XY	ZY	0	0	$1/\sqrt{2}$	0
XZ	ZZ	0	1	0	1

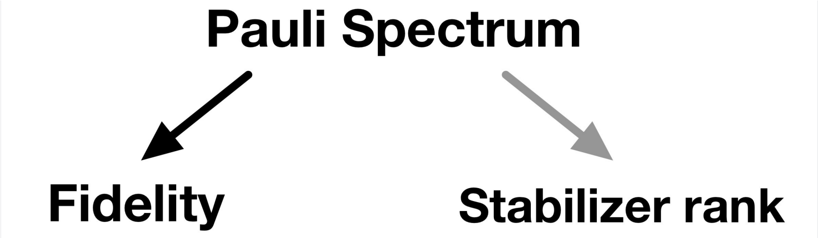
Stabilizer Rényi-α entropy: $H_\alpha = \frac{1}{1-\alpha} \log_2 \sum_{i=1}^{4^n} p_i^\alpha$

where $p_i = \frac{1}{2^n} \left(\frac{\langle\psi|P_i|\psi\rangle}{\langle\psi|\psi\rangle} \right)^2$

Measures of Magic



Pauli spectrum $\Longleftrightarrow$ Stabilizer fidelity



Product state

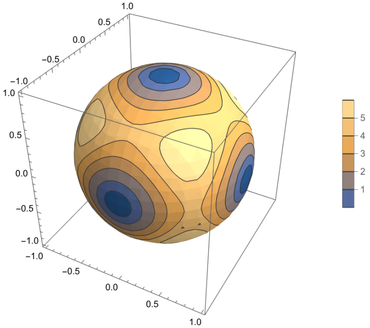


Figure: Rényi-2 entropy of the generalized state $G_{\theta\phi}^{\otimes 10}$

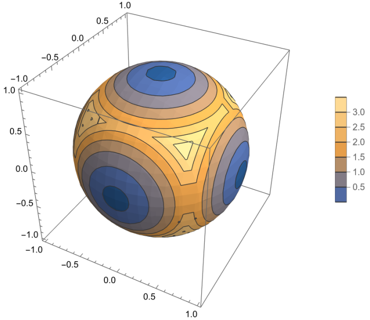


Figure: -log of the fidelity of the generalized state $G_{\theta\phi}^{\otimes 10}$

Pauli spectrum $\iff$ Stabilizer fidelity

Product state

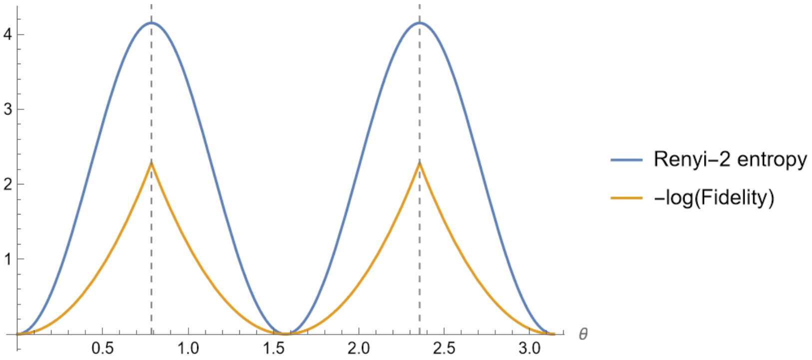


Figure: Rényi-2 entropy and -log of fidelity for $R_\theta^{\otimes 10}$

Pauli spectrum $\Longleftrightarrow$ Stabilizer fidelity

Entangled state

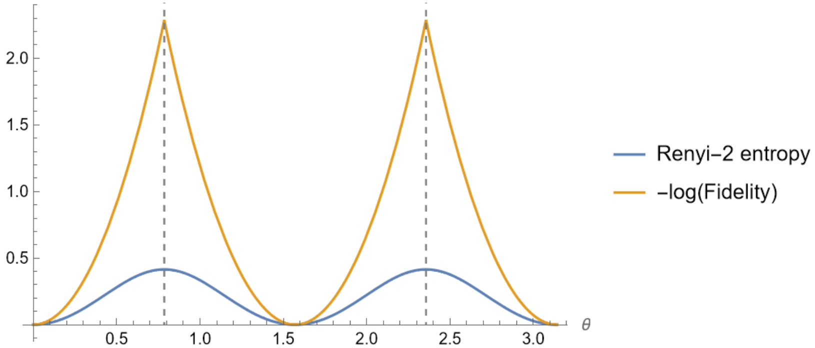
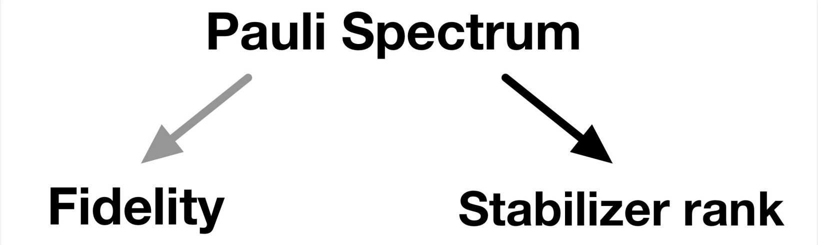


Figure: Rényi-2 entropy and -log of fidelity for $\frac{1}{\sqrt{2}}\left(|0\rangle^{\otimes 10} + e^{i\theta}|1\rangle^{\otimes 10}\right)$


Pauli Spectrum
Fidelity
Stabilizer rank

Pauli spectrum $\Longleftrightarrow$ Stabilizer rank

Can we say something about the stabilizer rank from the Pauli spectrum?

Pauli spectrum $\Longleftrightarrow$ Stabilizer rank

$$SR(|\phi\rangle) = \min\left\{\chi : |\phi\rangle = \sum_{i=1}^{\chi} c_i \, |\rho_i\rangle \right\}$$

For stabilizer rank 1 states, **YES**!

$\Downarrow$

Pauli spectrum of a stabilizer state only contains $1, 0, -1$.

Pauli spectrum $\iff$ Stabilizer rank

What about stabilizer rank 2 states?

Pauli spectrum $\Longleftrightarrow$ Stabilizer rank

$$SR(|\phi\rangle) = \min \left\{ \chi : |\phi\rangle = \sum_{i=1}^{\chi} c_i |\rho_i\rangle \right\}$$

Any stabilizer rank 2 state can be put in the canonical form by acting only Clifford operations C:

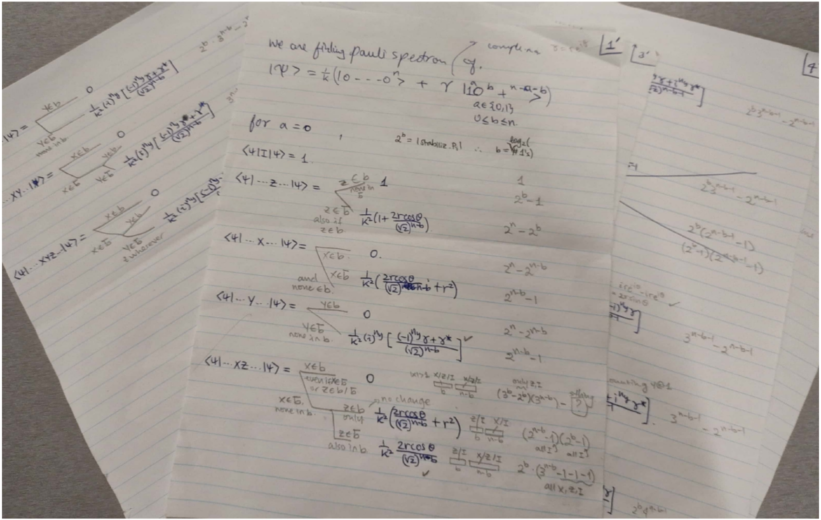
$$C|\phi\rangle = |\psi\rangle = \frac{1}{K}\left(|0\rangle^{\otimes n} + \gamma |1\rangle^{\otimes a} \otimes |0\rangle^{\otimes b} \otimes |+\rangle^{\otimes n-a-b}\right)$$

Pauli spectrum $\Longleftrightarrow$ Stabilizer rank

Clifford operations: take Pauli to Pauli

Clifford operations do **not** change the Pauli Spectrum!

Pauli spectrum ⟺ Stabilizer rank



Pauli spectrum $\Longleftrightarrow$ Stabilizer rank

Possible contraints:

1. Number of 0s
2. Number of 1s
3. Sum of all Pauli spectrum elements

Summary